Nonlinear matrix algebra and engineering applications

PART 2 : POLYNOMIAL FORM MATRICES (*)

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ABSTRACT

A matrix vector formalism is developed for systematizing the manipulation of sets of nonlinear algebraic equations. In this formalism all manipulations are performed by multiplication with specially constructed transformation matrices. For many important classes of nonlinearities, algorithms based on this formalism are presented for rearranging a set of equations so that their solution may be obtained by numerically searching along a single variable. Theory developed proves that all solutions are obtained.

5. POLYNOMIAL FORM MATRICES

If a set of equations does not contain linear variables, or if linear matrix techniques have already been applied to eliminate those variables which are linear, it is often necessary to use polynomial form matrix techniques.

Whenever a set of equations can be written in polynomial form with respect to any variable, it is possible to eliminate that variable and thereby reduce by one the number of equations which must be solved simultaneously. In general it is possible to repeat the procedure until the original problem is reduced to that of solving a single equation in a single variable.

Section 5.1 explains the various polynomial form matrix operations which are possible, and Section 5.2 develops the concept of rank as applied to polynomial form matrices. Sections 5.3, 5.4, and 5.5 present three elimination techniques, each of which has special advantages. Section 5.6 develops the theory of constant polynomial form matrices, while Sections 5.7 and 5.8 deal with polynomial form matrices of one variable and multiple variables.

Polynomial form matrix techniques are of great importance, since they are generally applicable. This one technique can always reduce n equations in n variables, where the nonlinearities are multinomial only, to a single equation in a single variable. The technique should not be applied indiscriminately, however, since other methods suggested in this paper, when applicable, often involve less work.

5.1. Polynomial form matrix operations

Both row and column operations may be performed on polynomial form matrices as well as on linear form matrices. The manner of manipulation and the rules of equivalence, subordinance and dominance are unchanged. Row operations are of great value in manipulating polynomial form matrices; however, column operations are of little use.

One particular row operation, which is identical with the shifting of a row of a polynomial form matrix either to the left or to the right, is singled out for special attention. This row operation, called the row shifting operation, may be accomplished by premultiplying the polynomial form matrix with a diagonal transformation matrix. If the vector associated with the polynomial form matrix is composed of powers of x_k, i.e., the polynomial form matrix is with respect to x_k, and it is desired to shift the j^{th} row by p places, the appropriate diagonal transformation matrices are

$$\begin{array}{cc}\begin{bmatrix} 1 & & & & \\ & 1 & & \bigcirc & \\ & & x_k^{p} & & \\ & \bigcirc & & 1 & \\ & & & & 1 \end{bmatrix} & \begin{array}{l} \\ \\ j^{th} \text{ row} \\ \\ \\ \end{array}\end{array} \qquad \begin{bmatrix} 1 & & & & \\ & 1 & & \bigcirc & \\ & & x_k^{p} & & \\ & \bigcirc & & 1 & \\ & & & & 1 \end{bmatrix}^{-1}$$

Shifts j^{th} row left by p places   Shifts j^{th} row right by p places

After premultiplication by the matrix on the left (right) the x_k^p (x_k^{-p}) introduced as the common factor of the elements of the j^{th} row may be removed by shifting the j^{th} row left (right) by p places.

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Since the row shifting operation is often needed, the symbolic row shifting operators sjrp and sjlp are introduced.

Definition

The symbols sjrp and sjlp mean "<u>s</u>hift the j^{th} row to the <u>r</u>ight by <u>p</u> places", and "<u>s</u>hift the j^{th} row to the <u>l</u>eft by <u>p</u> places".

The row shifting operator sjrp is a subordinate operator, while the row shifting operator sjlp is a dominant operator. More explicitly, if G is any $n \times (m+1)$ polynomial form augmented matrix, then

$$G \underset{[x_k \tilde{=} 0]}{\supset} \text{sjrp} \cdot G \quad \text{and} \quad G \underset{[x_k \tilde{=} 0]}{\subset} \text{sjlp} \cdot G \tag{5.1-1}$$

by Theorems 3-1 and 3-2.

Example

The polynomial form augmented matrix of the following nonlinear equations

$$\left.\begin{aligned} g_1(y)x^3 + g_2(y)x^2 + g_3(y)x + g_4(y) &= 0 \\ g_2'(y)x^2 + g_3'(y)x \qquad &= 0 \end{aligned}\right]$$

is

$$G = \begin{bmatrix} g_1(y) & g_2(y) & g_3(y) & g_4(y) \\ 0 & g_2'(y) & g_3'(y) & 0 \end{bmatrix}$$

According to (5.1-1)

$$G \underset{[x \tilde{=} 0]}{\supset} s2r1 \cdot G = \begin{bmatrix} g_1(y) & g_2(y) & g_3(y) & g_4(y) \\ 0 & 0 & g_2'(y) & g_3'(y) \end{bmatrix}$$

$$G \underset{[x \tilde{=} 0]}{\subset} s2l1 \cdot G = \begin{bmatrix} g_1(y) & g_2(y) & g_3(y) & g_4(y) \\ g_2'(y) & g_3'(y) & 0 & 0 \end{bmatrix}$$

The row shifting operations indicated by sjlp and sjrp may be accomplished by premultiplication and postmultiplication with constant matrices as follows.

$$\text{sjlp} \cdot G = D_1 G + D_2 G L$$

$$\text{sjrp} \cdot G = D_1 G + D_2 G R$$

where

$$D_1 = \underset{n \times n}{\begin{bmatrix} 1 & & & & & \\ & \ddots & & & \bigcirc & \\ & & 1 & & & \\ & & & 0 & & \\ & \bigcirc & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}} \leftarrow j^{th}\text{row} \rightarrow \underset{n \times n}{\begin{bmatrix} 0 & & & & & \\ & \ddots & & & \bigcirc & \\ & & 0 & & & \\ & & & 1 & & \\ & \bigcirc & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}} = D_2$$

$$L = \underset{(m+1) \times (m+1)}{\begin{bmatrix} & & \bigcirc \\ 1 & & \\ & \ddots & \\ \bigcirc & & 1 \end{bmatrix}} \leftarrow (p+1)^{th}\text{row} \qquad R = L^T$$

5.2. Rank of polynomial form matrices

The concepts of unconditional rank and conditional rank are useful for checking solution sets, and also form the basis of singular elimination as applied to polynomial form matrices.

The definition of unconditional rank applied to linear form matrices is unchanged. However, the definition of conditional rank as applied to polynomial form matrices should be singled out for special attention.

Definition

Consider any polynomial form coefficient or augmented matrix with respect to x_k containing $(n-1)$ variables, say $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$. For each numerical set $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n$ of these $(n-1)$ variables, the conditional rank of the polynomial form matrix with respect to $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n$ is defined as the order of the largest square array whose determinant does not vanish, where the array is formed from the matrix after substituting these $(n-1)$ values.

The other difference between the rank concepts developed for linear form matrices and the rank concepts of polynomial form matrices centers about the row shifting operation. Use of the row shifting operation may not leave the rank of a polynomial form matrix invariant.

Example

The unconditional rank of the polynomial form matrix

$$\begin{bmatrix} x^2 & 2x & 3y & 0 \\ 0 & x^2 & 2x & 3y \end{bmatrix}$$

is two. Shifting either the first row to the right by one (s1r1), or the second row to the left by one (s2l1) reduces the unconditional rank to one.

All other row operations leave the unconditional and conditional rank of a polynomial form matrix invariant.

5.3. Square elimination

If a set of equations can be written in polynomial form with respect to any variable, it is possible to eliminate that variable by reducing the corresponding polynomial form augmented matrix to degree zero. Square elimination is the first of three techniques presented for accomplishing this reduction.

Square elimination is based on row and row shifting operations. The algorithm is explained in detail. Each step of the reduction is shown to be equivalent to a premultiplication by a transformation matrix. If at any stage of the reduction a non-singular transformation matrix cannot be constructed, the algorithm automatically terminates. In this case the unconditional rank of the resulting polynomial form augmented matrix is one, and infinitely many solution sets may exist, as explained in Section 5.8.

The technique of square elimination is mechanically similar to triangular elimination. Square elimination introduces columns of zeroes either on the left or on the right. The algorithm given below is based on square elimination from the left; extension to elimination from the right is obvious.

Algorithm

The algorithm for performing square elimination is as follows.

(1) A polynomial form augmented matrix which represents the set of n equations in n unknowns is written down. It is preferable to choose the polynomial form matrix of lowest degree. Let this matrix be with respect to x_k.

(2) Rearrange the rows such that the 1,1 element of the matrix becomes the simplest possible non-zero function, preferably a non-zero constant. Denote this polynomial form augmented matrix by $G^{[1]}$.

$$G^{[1]} = \begin{bmatrix} g^{[1]}_{11} & g^{[1]}_{12} & g^{[1]}_{13} & \cdots & g^{[1]}_{1\,m+1} \\ g^{[1]}_{21} & g^{[1]}_{22} & g^{[1]}_{23} & \cdots & g^{[1]}_{2\,m+1} \\ g^{[1]}_{31} & g^{[1]}_{32} & g^{[1]}_{33} & \cdots & g^{[1]}_{3\,m+1} \\ \vdots & \vdots & \vdots & & \vdots \\ g^{[1]}_{n1} & g^{[1]}_{n2} & g^{[1]}_{n3} & \cdots & g^{[1]}_{n\,m+1} \end{bmatrix} \quad (5.3\text{-}1)$$

(3) Premultiplying (5.3-1) by the following transformation matrix

$$\begin{bmatrix} 1 & & & \\ -g^{[1]}_{21} & g^{[1]}_{11} & & \bigcirc \\ -g^{[1]}_{31} & & \ddots & \\ \vdots & \bigcirc & & \\ -g^{[1]}_{n1} & & & g^{[1]}_{11} \end{bmatrix} \quad (5.3\text{-}2)$$

produces the result

$$\begin{bmatrix} 1 & & & \\ -g^{[1]}_{21} & g^{[1]}_{11} & & \bigcirc \\ -g^{[1]}_{31} & & \ddots & \\ \vdots & \bigcirc & & \\ -g^{[1]}_{n1} & & & g^{[1]}_{11} \end{bmatrix} \begin{bmatrix} g^{[1]}_{11} & g^{[1]}_{12} & g^{[1]}_{13} & \cdots & g^{[1]}_{1\,m+1} \\ g^{[1]}_{21} & g^{[1]}_{22} & g^{[1]}_{23} & \cdots & g^{[1]}_{2\,m+1} \\ g^{[1]}_{31} & g^{[1]}_{32} & g^{[1]}_{33} & \cdots & g^{[1]}_{3\,m+1} \\ \vdots & \vdots & \vdots & & \vdots \\ g^{[1]}_{n1} & g^{[1]}_{n2} & g^{[1]}_{n3} & \cdots & g^{[1]}_{n\,m+1} \end{bmatrix}$$

$$\begin{bmatrix} g^{[1]}_{11} & g^{[1]}_{12} & g^{[1]}_{13} & \cdots & g^{[1]}_{1\,m+1} \\ 0 & g'_{22} & g'_{23} & \cdots & g'_{2\,m+1} \\ \downarrow & g'_{32} & g'_{33} & \cdots & g'_{3\,m+1} \\ & \vdots & \vdots & & \vdots \\ 0 & g'_{n2} & g'_{n3} & \cdots & g'_{n\,m+1} \end{bmatrix} \quad (5.3\text{-}3)$$

where (5.3-1) $\underset{[g^{[1]}_{11}=0]}{\underset{\sim}{\subset}}$ (5.3-3) by Theorem 3-1.

(4) Rearrange the 2nd through nth rows of the matrix on the right hand side of (5.3-3) such that the 2,2 element becomes the simplest possible non-zero function, preferably a non-zero constant. Denote this matrix by $G^{[2]'}$.

$$G^{[2]'} = \begin{bmatrix} g^{[1]}_{11} & g^{[1]}_{12} & g^{[1]}_{13} & \cdots & g^{[1]}_{1\,m+1} \\ 0 & g^{[2]}_{22} & g^{[2]}_{23} & \cdots & g^{[2]}_{2\,m+1} \\ \downarrow & g^{[2]}_{32} & g^{[2]}_{33} & \cdots & g^{[2]}_{3\,m+1} \\ & \vdots & \vdots & & \vdots \\ 0 & g^{[2]}_{n2} & g^{[2]}_{n3} & & g^{[2]}_{n\,m+1} \end{bmatrix} \quad (5.3\text{-}4)$$

(5) Operating on the left of (5.3-4) with

$$\text{s2r1}\begin{bmatrix} -g^{[2]}_{22} & g^{[1]}_{11} & & \bigcirc \\ & 1 & & \\ \bigcirc & & \ddots & \\ & & & 1 \end{bmatrix}\text{s2l1} \quad (5.3\text{-}5)$$

yields

$$G^{[2]'} \underset{[g^{[2]}_{22}=0]}{\underset{\sim}{\subset}} G^{[2]} = \begin{bmatrix} 0 & g^{[2]}_{12} & g^{[2]}_{13} & \cdots & g^{[2]}_{1\,m+1} \\ & g^{[2]}_{22} & g^{[2]}_{23} & \cdots & g^{[2]}_{2\,m+1} \\ \downarrow & g^{[2]}_{32} & g^{[2]}_{33} & \cdots & g^{[2]}_{3\,m+1} \\ & \vdots & \vdots & & \vdots \\ 0 & g^{[2]}_{n2} & g^{[2]}_{n3} & \cdots & g^{[2]}_{n\,m+1} \end{bmatrix} \quad (5.3\text{-}6)$$

Remark

The operator (5.3-5) is a dominant operator, since it is equivalent to the operator

$$\begin{bmatrix} -g^{[2]}_{22} & x_k g^{[1]}_{11} & & \bigcirc \\ & 1 & & \\ \bigcirc & & \ddots & \\ & & & 1 \end{bmatrix}$$

Any additional solution sets, if they exist, must satisfy $g_{22}^{[2]} = 0$.

(6) Premultiplying $G^{[2]}$ by the transformation matrix

$$\begin{bmatrix} g_{22}^{[2]} & -g_{12}^{[2]} & & \bigcirc \\ 0 & 1 & & \\ \downarrow & -g_{32}^{[2]} & g_{22}^{[2]} & \\ & \vdots & \bigcirc & \searrow \\ 0 & -g_{n2}^{[2]} & & g_{22}^{[2]} \end{bmatrix} \qquad (5.3\text{-}7)$$

produces the result

$$\begin{bmatrix} g_{22}^{[2]} & -g_{12}^{[2]} & & \bigcirc \\ 0 & 1 & & \\ \downarrow & -g_{32}^{[2]} & g_{22}^{[2]} & \\ & \vdots & \bigcirc & \searrow \\ 0 & -g_{n2}^{[2]} & & g_{22}^{[2]} \end{bmatrix} \begin{bmatrix} 0 & g_{12}^{[2]} & g_{13}^{[2]} \cdots g_{1\,m+1}^{[2]} \\ & g_{22}^{[2]} & g_{23}^{[2]} \cdots g_{2\,m+1}^{[2]} \\ & g_{32}^{[2]} & g_{33}^{[2]} \cdots g_{3\,m+1}^{[2]} \\ \downarrow & \vdots & \vdots \qquad \vdots \\ 0 & g_{n2}^{[2]} & g_{n3}^{[2]} \cdots g_{n\,m+1}^{[2]} \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & g_{13}'' \cdots g_{1\,m+1}'' \\ & g_{22}^{[2]} & g_{23}^{[2]} \cdots g_{2\,m+1}^{[2]} \\ & 0 & g_{33}'' \cdots g_{3\,m+1}'' \\ \downarrow & \downarrow & \vdots \qquad \vdots \\ 0 & 0 & g_{n3}'' \cdots g_{n\,m+1}'' \end{bmatrix} \qquad (5.3\text{-}8)$$

(7) Rearrange the rows, excluding the second row, of the matrix on the right hand side of (5.3-8) such that the 3,3 element becomes the simplest possible non-zero function, preferably a constant. Denote this matrix by $G^{[3]'}$.

$$G^{[3]'} = \begin{bmatrix} 0 & 0 & g_{13}^{[3]} \cdots g_{1\,m+1}^{[3]} \\ & g_{22}^{[2]} & g_{23}^{[2]} \cdots g_{2\,m+1}^{[2]} \\ & 0 & g_{33}^{[3]} \cdots g_{3\,m+1}^{[3]} \\ \downarrow & \downarrow & \vdots \qquad \vdots \\ 0 & 0 & g_{n3}^{[3]} \cdots g_{n\,m+1}^{[3]} \end{bmatrix} \qquad (5.3\text{-}9)$$

(8) Operating on the left of (5.3-9) with

$$s3r1 \begin{bmatrix} 1 & & & \bigcirc \\ & -g_{33}^{[3]} & g_{22}^{[2]} & \\ \bigcirc & & 1 \searrow & \\ & & & 1 \end{bmatrix} s3l1 \qquad (5.3\text{-}10)$$

yields

$$G^{[3]'} \underset{[g_{33}^{[3]}=0]}{\underset{\sim}{\subset}} G^{[3]} = \begin{bmatrix} 0 & 0 & g_{13}^{[3]} \cdots g_{1\,m+1}^{[3]} \\ & & g_{23}^{[3]} \cdots g_{2\,m+1}^{[3]} \\ & & g_{33}^{[3]} \cdots g_{3\,m+1}^{[3]} \\ \downarrow & \downarrow & \vdots \qquad \vdots \\ 0 & 0 & g_{n3}^{[3]} \cdots g_{n\,m+1}^{[3]} \end{bmatrix} \qquad (5.3\text{-}11)$$

(9) Continuing in this manner, it is possible to transform the original matrix (5.3-1) into a form where x_k has been completely eliminated from n−1 equations and only appears linearly in one equation. Thus

$$\begin{bmatrix} g_{11}^{[1]} & g_{12}^{[1]} & g_{13}^{[1]} \cdots g_{1\,m+1}^{[1]} \\ g_{21}^{[1]} & g_{22}^{[1]} & g_{23}^{[1]} \cdots g_{2\,m+1}^{[1]} \\ \vdots & \vdots & \vdots \qquad \vdots \\ g_{n1}^{[1]} & g_{n2}^{[1]} & g_{n3}^{[1]} \cdots g_{n\,m+1}^{[1]} \end{bmatrix}$$

$$\underset{[g_{11}^{[1]} g_{22}^{[2]} \cdots g_{lm}^{[m]} = 0]}{\underset{\sim}{\subset}} \begin{bmatrix} & 0 & g_{1\,m+1}^{[m+1]} \\ & \downarrow & \vdots \\ & 0 & g_{l-1\,m+1}^{[m+1]} \\ \bigcirc & g_{lm}^{m} & g_{lm+1}^{[m]} \\ & & g_{l+1m+1}^{[m+1]} \\ & \downarrow & \vdots \\ & 0 & g_{nm+1}^{[m+1]} \end{bmatrix} \qquad (5.3\text{-}12)$$

Any additional solution sets, if they exist, must satisfy $g_{11}^{[1]} g_{22}^{[2]} \cdots g_{lm}^{[m]} = 0$. It should be noted that when $n > m$, steps (6), (7) and (8) may be applied repeatedly to obtain (5.3-12). However, when $n < m$, it is necessary to return to step (2) after n columns of zeroes have been introduced by the repeated application of steps (6), (7) and (8). The value of the subscript l is determined by dividing m by n to yield an integral quotient plus a remainder. The remainder is l.

(10) The n-1 equations which do not contain x_k, namely all of the equations on the right hand side of (5.3-12), may be further reduced by repeated application of this algorithm. If the original equations are multinomial with integral exponents it will be possible to reduce the original n equations in n variables to one equation in a single variable.
The algorithm for performing square elimination on the left may be modified and used to perform elimination on the right. It is generally advantageous to operate on the side which contains the simplest functions, preferably constants. Whenever the most complicated functions appear in the center of the matrix, it is generally advantageous to operate on both the left and right hand sides.

Example

The three equations in three unknowns :

$$\begin{bmatrix} x_2 x_1^2 + x_3 x_1 + 1 = 0 \\ x_2^2 x_1^2 + x_3^2 x_1 - 1 = 0 \\ x_2^3 x_1^2 + x_3^3 x_1 + 1 = 0 \end{bmatrix} \tag{5.3-13}$$

are associated with the polynomial form augmented matrix of lowest degree with respect to x_1.

$$\begin{bmatrix} x_2 & x_3 & 1 \\ x_2^2 & x_3^2 & -1 \\ x_2^3 & x_3^3 & 1 \end{bmatrix} \tag{5.3-14}$$

First, square elimination from the right will be used, since elements of the last column of the matrix are non-zero constants. Premultiplying (5.3-14) by a transformation matrix yields

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_2 & x_3 & 1 \\ x_2^2 & x_3^2 & -1 \\ x_2^3 & x_3^2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} x_2 & x_3 & 1 \\ x_2(1+x_2) & x_3(1+x_3) & 0 \\ x_2(1-x_2^2) & x_3(1-x_3^2) & 0 \end{bmatrix} \tag{5.3-15}$$

Second, operating on the right side of (5.3-15) with

$$S_2 l_1 \begin{bmatrix} -x_3(1+x_3) & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S_2 r_1$$

yields

$$\begin{bmatrix} x_2 & x_3 & 1 \\ x_2(1+x_2) & x_3(1+x_3) & 0 \\ x_2(1-x_2^2) & x_3(1-x_3^2) & 0 \end{bmatrix}$$

$$\underset{[x_3(1+x_3)=0]}{\underset{\sim}{\subseteq}} \begin{bmatrix} -x_2x_3(1+x_3) & x_2(1+x_2)-x_3^2(1+x_3) & 0 \\ x_2(1+x_2) & x_3(1+x_3) & 0 \\ x_2(1-x_2^2) & x_3(1-x_3^2) & 0 \end{bmatrix} \tag{5.3-16}$$

All rows of the right side of (5.3-16) are then shifted one place to the right

$$(5.3\text{-}16) \underset{[x_1=0]}{\underset{\sim}{\supseteq}} \begin{bmatrix} 0 & -x_2x_3(1+x_3) & x_2(1+x_2)-x_3^2(1+x_3) \\ 0 & x_2(1+x_2) & x_3(1+x_3) \\ 0 & x_2(1-x_2^2) & x_3(1-x_3^2) \end{bmatrix} \tag{5.3-17}$$

Third, square elimination from the left will be applied by premultiplying (5.3-17) by a transformation matrix :

$$\begin{bmatrix} 1 & 0 & 0 \\ (1+x_2) & x_3(1+x_3) & 0 \\ (1-x_2^2) & 0 & x_3(1+x_3) \end{bmatrix} \begin{bmatrix} -x_2x_3(1+x_3) & x_2(1+x_2)-x_3^2(1+x_3) \\ x_2(1+x_2) & x_3(1+x_3) \\ x_2(1-x_2^2) & x_3(1-x_3^2) \end{bmatrix}$$

$$\underset{[x_3(1+x_3)=0]}{\underset{\sim}{\subseteq}} \begin{bmatrix} -x_2x_3(1+x_3) & x_2(1+x_2)-x_3^2(1+x_3) \\ 0 & (1+x_2)[x_2(1+x_2)-x_3^2(1+x_3)]+x_3^2(1+x_3)^2 \\ 0 & (1-x_2^2)[x_2(1+x_2)-x_3^2(1+x_3)]+x_3^2(1+x_3)(1-x_3^2) \end{bmatrix} \tag{5.3-18}$$

Fourth, premultiplying the right side of (5.3-18) by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & (1-x_2) & -1 \end{bmatrix}$$

yields the equivalent matrix

$$\begin{bmatrix} -x_2x_3(1+x_3) & x_2(1+x_2)-x_3^2(1+x_3) \\ 0 & (1+x_2)[x_2(1+x_2)-x_3^2(1+x_3)] \\ & \quad + x_3^2(1+x_3)^2 \\ 0 & x_3^2(1+x_3)^2\ (x_3-x_2) \end{bmatrix} \tag{5.3-19}$$

From the last row of (5.3-19), we obtain the possible solutions :

$$\begin{bmatrix} x_3 = 0 \\ x_3 = -1 \\ x_3 = x_2 \end{bmatrix} \tag{5.3-20}$$

Substituting x_3 value in (5.3-20) into the second row of (5.3-19) yields four possible additional solution sets :

$$\begin{bmatrix} x_3 = 0, \ x_2 = 0 \\ x_3 = 0, \ x_2 = -1 \\ x_3 = -1, \ x_2 = 0 \\ x_3 = -1, \ x_2 = -1 \end{bmatrix} \tag{5.3-21}$$

Fifth, investigate whether solutions in (5.3-21) are additional solutions; substituting (5.3-21) into the right hand side of (5.3-15) yields four constant polynomial form matrices of lowest degree which have rank one (see section 5.6);

(For $x_3 = 0, x_2 = 0$), (For $x_3 = 0, x_2 = -1$),

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

(For $x_3 = -1, x_2 = 0$), (For $x_3 = -1, x_2 = -1$)

$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.3-22}$$

Since the lowest degree of the first matrix of (5.3-22) is zero, according to Theorem 5-2 the set $x_3 = 0$ and $x_2 = 0$ is, therefore, considered an additional solution. Based on (5.3-22) and Theorem 5-2, the original problem (5.3-14) has five solution sets

$$\begin{bmatrix} x_3 = 0, \ x_2 = -1, \ x_1 = \pm 1 \\ x_3 = -1, \ x_2 = 0, \ x_1 = 1 \\ x_3 = -1, \ x_2 = -1, \ x_1 = \dfrac{-1 \pm \sqrt{5}}{2} \end{bmatrix} \tag{5.3-23}$$

5.4. Cross elimination

Cross elimination is an elimination technique similar to square elimination. This technique can also reduce the degree of polynomial form matrices. It has the advantage of being mechanically simpler than square elimination, but generally introduces more additional solution sets.

Algorithm

The algorithm for performing cross elimination is as follows.

(1) A polynomial form augmented matrix which represents the set of n equations in n unknowns is written down. It is preferable to choose the polynomial form matrix of lowest degree. Let this matrix be with respect to x_k.

(2) Rearrange the rows in such a way that the element 1,1 and the determinant formed by the element 1,1; 1, m + 1; 2,1 and 2, m + 1, i.e.,

$$\begin{bmatrix} (1,1) & (1,m+1) \\ (2,1) & (2,m+1) \end{bmatrix}$$

become the simplest possible non-zero functions, preferably constants. Denote this polynomial form matrix by $G^{[1]}$.

$$G^{[1]} = \begin{bmatrix} g_{11}^{[1]} & g_{12}^{[1]} & \cdots & g_{1\,m}^{[1]} & g_{1\,m+1}^{[1]} \\ g_{21}^{[1]} & g_{22}^{[1]} & \cdots & g_{2\,m}^{[1]} & g_{2\,m+1}^{[1]} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{n1}^{[1]} & g_{n2}^{[1]} & \cdots & g_{n\,m}^{[1]} & g_{n\,m+1}^{[1]} \end{bmatrix} \tag{5.4-1}$$

(3) Premultiplying (5.4-1) by the following transformation matrix

$$\begin{bmatrix} -g_{2\,m+1}^{[1]} & g_{1\,m+1}^{[1]} & & \\ -g_{21}^{[1]} & g_{11}^{[1]} & & \bigcirc \\ \vdots & & \ddots & \\ -g_{n\,1}^{[1]} & \bigcirc & & g_{11}^{[1]} \end{bmatrix} \tag{5.4-2}$$

produces the result

$$
\begin{bmatrix}
-g^{[1]}_{2\,m+1} & g^{[1]}_{1\,m+1} & & \\
-g^{[1]}_{21} & g^{[1]}_{11} & & \bigcirc \\
\vdots & & \ddots & \\
-g^{[1]}_{n\,1} & \bigcirc & & g^{[1]}_{11}
\end{bmatrix}
\begin{bmatrix}
g^{[1]}_{11} & g^{[1]}_{12} & \cdots & g^{[1]}_{1\,m} & g^{[1]}_{1\,m+1} \\
g^{[1]}_{21} & g^{[1]}_{22} & \cdots & g^{[1]}_{2\,m} & g^{[1]}_{2\,m+1} \\
\vdots & \vdots & & \vdots & \vdots \\
g^{[1]}_{n\,1} & g^{[1]}_{n\,2} & \cdots & g^{[1]}_{n\,m} & g^{[1]}_{n\,m+1}
\end{bmatrix}
$$

$$
\sim
\begin{bmatrix}
g^{[2]'}_{12} & g^{[2]'}_{13} & \cdots & g^{[2]'}_{1\,m+1} & 0 \\
0 & g^{[2]'}_{22} & \cdots & g^{[2]'}_{2\,m} & g^{[2]'}_{2\,m+1} \\
\downarrow & \vdots & & \vdots & \vdots \\
0 & g^{[2]'}_{n\,2} & \cdots & g^{[2]'}_{n\,m} & g^{[2]'}_{n\,m+1}
\end{bmatrix}
\qquad (5.4\text{-}3)
$$

where $(5.4\text{-}1) \underset{[g^{[1]}_{11}\, g^{[2]'}_{12} = 0]}{\overset{\subset}{\sim}} (5.4\text{-}3)$.

The condition under which additional solution sets may exist, namely $g^{[1]}_{11}\, g^{[2]'}_{12} = 0$, where $g^{[2]'}_{12} = g^{[1]}_{21}\, g^{[1]}_{1\,m+1} - g^{[1]}_{11}\, g^{[1]}_{2\,m+1}$, derives from the determinant of (5.4-2) and Theorem 3-1.

(4) Shift the first row of the matrix on the right hand side of (5.4-3) right by one place, i.e., apply the operator s1r1. Rearrange the rows such that the element 1,2 and the determinant formed by the elements 1,2; 1,m+1; 2,2 and 2,m+1, i.e.,

$$
\begin{vmatrix}
(1,2) & (1, m+1) \\
(2,2) & (2, m+1)
\end{vmatrix}
$$

become the simplest possible non-zero functions, preferably non-zero constants. Denote this polynomial form matrix by $G^{[2]}$.

$$
G^{[2]} =
\begin{bmatrix}
0 & g^{[2]}_{12} & \cdots & g^{[2]}_{1\,m} & g^{[2]}_{1\,m+1} \\
 & g^{[2]}_{22} & \cdots & g^{[2]}_{2\,m} & g^{[2]}_{2\,m+1} \\
\downarrow & \vdots & & \vdots & \vdots \\
0 & g^{[2]}_{n\,2} & \cdots & g^{[2]}_{n\,m} & g^{[2]}_{n\,m+1}
\end{bmatrix}
\qquad (5.4\text{-}4)
$$

The statement of equivalence relating $G^{[1]}$ and $G^{[2]}$, according to Theorem 3-1 and Section 5.1, is

$$
G^{[1]} \underset{[g^{[1]}_{11}\, g^{[2]'}_{12} = 0]}{\overset{\subset}{\sim}} (5.4\text{-}3) \underset{[x_k = 0]}{\overset{\supset}{\sim}} G^{[2]} \qquad (5.4\text{-}5)
$$

where

$$
g^{[2]'}_{12} = g^{[1]}_{21}\, g^{[1]}_{1\,m+1} - g^{[1]}_{11}\, g^{[1]}_{2\,m+1}.
$$

(5) Continuing in this manner until all but the last two columns are zero yields

$$
G^{[m]} =
\begin{bmatrix}
 & g^{[m]}_{1\,m} & g^{[m]}_{1\,m+1} \\
\bigcirc & g^{[m]}_{2\,m} & g^{[m]}_{2\,m+1} \\
 & \vdots & \vdots \\
 & g^{[m]}_{n\,m} & g^{[m]}_{n\,m+1}
\end{bmatrix}
\qquad (5.4\text{-}6)
$$

where the rows have already been rearranged such that $g^{[m]}_{1\,m}$ is the simplest possible non-zero function, preferably a constant.

(6) Applying the technique of square elimination by multiplying through by a transformation matrix completes the problem.

$$
\begin{bmatrix}
1 & & & \\
-g^{[m]}_{2\,m} & g^{[m]}_{1\,m} & & \bigcirc \\
\vdots & & \ddots & \\
-g^{[m]}_{n\,m} & \bigcirc & & g^{[m]}_{1\,m}
\end{bmatrix}
\begin{bmatrix}
 & g^{[m]}_{1\,m} & g^{[m]}_{1\,m+1} \\
\bigcirc & g^{[m]}_{2\,m} & g^{[m]}_{2\,m+1} \\
 & \vdots & \vdots \\
 & g^{[m]}_{n\,m} & g^{[m]}_{n\,m+1}
\end{bmatrix}
$$

$$
\begin{bmatrix}
 & g^{[m]}_{1\,m} & g^{[m]}_{1\,m+1} \\
\bigcirc & 0 & g^{[m+1]'}_{2\,m+1} \\
 & \downarrow & \vdots \\
 & 0 & g^{[m+1]'}_{n\,m+1}
\end{bmatrix}
\qquad (5.4\text{-}7)
$$

The final statement of equivalence is

$$
\begin{cases}
(5.4\text{-}1) \underset{[x_k = 0]}{\overset{\supset}{\sim}} (5.4\text{-}7) \\[2ex]
(5.4\text{-}1) \underset{[(g^{[1]}_{11}\, g^{[2]'}_{12})(g^{[2]}_{12}\, g^{[3]'}_{13}) \cdots (g^{[m-1]}_{1\,m-1}\, g^{[m]'}_{1\,m})\, g^{[m]}_{1\,m} = 0}{\overset{\subset}{\sim}} (5.4\text{-}7
\end{cases}
\qquad (5.4\text{-}8)
$$

i.e., any missing solution sets, if they exist, must satisfy $x_k = 0$ and additional solution sets, if they exist, must satisfy

$$
(g^{[1]}_{11}\, g^{[2]'}_{12})(g^{[2]}_{12}\, g^{[3]'}_{13}) \cdots (g^{[m-1]}_{1\,m-1}\, g^{[m]'}_{1\,m})\, g^{[m]}_{1\,m} = 0
$$

This algorithm performs cross elimination on the left. It may readily be modified to perform elimination on the right, or even mixed elimination. Mixed elimination steps, i.e., introduction of several zeroes on both the left and right, are usually not desirable since they generally introduce more additional solution sets than either left or right eliminations.

Example

The same problem solved by square elimination in Section 5.3

$$\begin{bmatrix} x_2 & x_3 & 1 \\ x_2^2 & x_3^2 & -1 \\ x_2^3 & x_3^3 & 1 \end{bmatrix} \tag{5.4-9}$$

is now treated by cross elimination. First, premultiplying (5.4-9) by a transformation matrix produce the result

$$\begin{bmatrix} -x_2 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 & x_3 & 1 \\ x_2^2 & x_3^2 & -1 \\ x_2^3 & x_3^3 & 1 \end{bmatrix}$$

$$\underset{[-(x_2+1)=0]}{\underset{\sim}{\subset}} \begin{bmatrix} 0 & -x_3(x_2-x_3)-(x_2+1) \\ x_2(x_2+1) & x_3(x_3+1) & 0 \\ x_2(x_2^2-1) & x_3(x_3^2-1) & 0 \end{bmatrix} \tag{5.4-10}$$

Second, shifting the second and third rows of the matrix on the right hand side of (5.14-10) right by one place and changing the sign of the first row yield

$$(5.4\text{-}10) \underset{[x_1=0]}{\underset{\sim}{\supset}} \begin{bmatrix} x_3(x_2-x_3) & (x_2+1) \\ x_2(x_2+1) & x_3(x_3+1) \\ x_2(x_2^2-1) & x_3(x_3^2-1) \end{bmatrix} \tag{5.4-11}$$

Third, the order of the rows of the right side of (5.4-11) is changed to place the second row first, and then the technique of square elimination will be applied. This yields the following result

$$\begin{bmatrix} 1 & 0 & 0 \\ x_3(x_2-x_3) & -x_2(x_2+1) & 0 \\ (x_2-1) & 0 & -1 \end{bmatrix} \begin{bmatrix} x_2(x_2+1) & x_3(x_3+1) \\ x_3(x_2-x_3) & (x_2+1) \\ x_2(x_2^2-1) & x_3(x_3^2-1) \end{bmatrix}$$

$$\underset{[x_2(x_2+1)=0]}{\underset{\sim}{\subset}} \begin{bmatrix} x_2(x_2+1) & x_3(x_3+1) \\ 0 & x_3^2(x_3+1)(x_2-x_3)-x_2(x_2+1)^2 \\ 0 & x_3(x_3+1)(x_2-x_3) \end{bmatrix} \tag{5.4-12}$$

Fourth, solving the last row of (5.4-12) for x_3 and substituting the x_3 values into the second row of (5.4-12) yields the same four possible additional solution sets as (5.3-21).

Fifth, substituting (5.3-21) into (5.4-9) yields four constant polynomial form matrices. Each of those matrices is equivalent to the matrix of (5.3-23) for the same values of x_2 and x_3 used (see Section 5.6). Thus the same solution sets

$$\left[\begin{array}{lll} x_3 = 0, & x_2 = -1, & x_1 = \pm 1 \\ x_3 = -1, & x_2 = 0, & x_1 = 1 \\ x_3 = -1, & x_2 = -1, & x_2 = \dfrac{-1 \pm \sqrt{5}}{2} \end{array}\right. \tag{5.4-13}$$

as (5.3-22) are obtained.

5.5. Singular elimination

Occasionally singular elimination is a valuable short cut for solving polynomial form matrices. The technique is identical with that described in Section 4.4 for linear form matrices.

Briefly, if a polynomial form augmented matrix of dimension $n \times (m+1)$ is such that $n > (m+1)$, the rows of the $n \times m$ polynomial form coefficient matrix must be nonlinearly dependent by Theorem 4-2, and a set of $\emptyset$'s may be found. If $n < (m+1)$, two cases arise. If the unconditional rank of the $n \times m$ polynomial form coefficient matrix is less than n, the row vectors are also nonlinearly dependent and a set of $\emptyset$'s may be found. If the unconditional rank is equal to n, the row vectors of the polynomial form coefficient matrix are nonlinearly independent and it is necessary to reduce the number of non-zero columns by some technique such as square or cross elimination until the unconditional rank is less than n.

Example

The same problem solved by square and cross elimination

$$\begin{bmatrix} x_2 & x_3 & 1 \\ x_2^2 & x_3^2 & -1 \\ x_2^3 & x_3^2 & 1 \end{bmatrix} \tag{5.5-1}$$

is now treated by singular elimination. Since the number of rows n = 3, and the unconditional rank of the coefficient matrix is 2, a set of $\emptyset$'s exists and may be found as shown in Section 4.4.

$$\emptyset_1 = -x_2 x_3 \quad \emptyset_2 = x_2 + x_3 \quad \emptyset_3 = -1 \tag{5.5-2}$$

Premultiplying (5.5-1) by the appropriate transformation matrix yields

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_2x_3 & (x_2+x_3) & -1 \end{bmatrix} \begin{bmatrix} x_2 & x_3 & 1 \\ x_2^2 & x_3^2 & -1 \\ x_3^3 & x_3^3 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} x_2 & x_3 & 1 \\ x_2^2 & x_3^2 & -1 \\ 0 & 0 & -(x_2+1)(x_3+1) \end{bmatrix} \tag{5.5-3}$$

The last row of (5.5-3), i.e., $-(x_2+1)(x_3+1)=0$, indicates that solution sets exist only if $x_2=-1$ or $x_3=-1$. Substituting $x_2=-1$ into (5.5-3) yields

$$\begin{bmatrix} -1 & x_3 & 1 \\ 1 & x_3^2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.5-4}$$

Clearly (5.5-4) has common solution sets if and only if x_3 is chosen so that the rank of (5.5-4) becomes one. By observation, $x_3=0$, or $x_3=-1$ satisfies this requirement. Thus the solution sets corresponding to $x_2=-1$ are

$$\left[\begin{array}{lll} x_2=-1 & x_3=0 & x_1=\pm 1 \\ x_2=-1 & x_3=-1 & x_1=\dfrac{-1\pm\sqrt{5}}{2} \end{array}\right. \tag{5.5-5}$$

Similarly, substituting $x_3=-1$ into (5.5-3) yields

$$\begin{bmatrix} x_2 & -1 & 1 \\ x_2^2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.5-6}$$

and by observation $x_2=0$ or $x_2=-1$ can reduce the rank of (5.5-6) to one. The solution sets corresponding to $x_3=-1$ are

$$\begin{array}{lll} x_2=0 & x_3=-1 & x_1=1 \\ x_2=-1 & x_3=-1 & x_1=\dfrac{-1\pm\sqrt{5}}{2} \end{array}$$

Thus the same solution sets as (5.3-23) are obtained.

5.6. Constant polynomial form matrices

The problem of finding the common roots of a set of n non-linear polynomial equations in one unknown x can be solved readily by use of the polynomial form matrix techniques already presented. Various classical methods of solving this problem have been known for some time. These include the highest common divisor method [1], Bezout's method [2], and Sylvester's determinant [1]. The new technique is much faster than the classical techniques. This section provides the basis of testing possible solution sets in more complicated cases.

Definition

A polynomial matrix is said to be of lowest degree if and only if each of its rows has been shifted to the right as far as possible.

Definition

A constant polynomial form matrix is one which has only constant elements.
Clearly, a constant polynomial matrix with respect to x of lowest degree does not have the common trivial solution $x=0$.

Theorem 5-1

Any constant polynomial form augmented matrix A_r^m of lowest degree $\underline{m}$ and rank $\underline{r}$ can be reduced by row and row shifting operations to an equivalent matrix A_1^d of lowest degree $\underline{d}$ which has a rank of $\underline{\text{one}}$. In symbolic notation

$$A_r^m \sim A_1^d$$

Proof

The proof is in two parts. First it is established that the rank can always be reduced to one; second it is shown that equivalence is maintained.
If the rank of the matrix is not one, there must exist at least two non-zero rows. Row and row shifting operations can then be continued until all but one row is zero. Thus the rank can always be reduced to one.
By Theorem 3-1 and Section 5.1, row and row shifting operations are associated with the statement of equivalence

$$A_r^m \underset{[x=0]}{\subseteq} A_1^d \quad \text{or} \quad A_r^m \underset{[x=0]}{\supsetapprox} A_1^d$$

Since both A_r^m and A_1^d are of lowest degree, the possible additional or missing trivial solution $x=0$ does not satisfy either A_r^m or A_1^d. Thus

$$A_r^m \sim A_1^d$$

Theorem 5-2

Disregarding the possible trivial solution $x=0$, a set of polynomial equations in one unknown x have common complex roots if and only if their associated reduced matrix A_1^d has a lowest degree $d>0$. The number of common complex roots is d.

Proof

By Theorem 5-1, $A_r^m \sim A_1^d$. Since the rank of A_1^d is one, this implies that A_1^d is equivalent to a single polynomial equation of degree d, which has of course exactly d complex roots. Since equivalence has been maintained throughout, these d roots must be precisely the set of common roots of A_r^m.

Example

Find the common solutions which satisfy the three polynomial equations in one variable x

$$
\left.\begin{array}{l}
f_1 = x^6 + 2x^5 + x^3 + 3x^2 + 3x + 2 = 0 \\
f_2 = x^4 + 4x^3 + 4x^2 - x - 2 = 0 \\
f_3 \quad x^3 + 2x^2 - x - 2 = 0
\end{array}\right] \quad (5.6\text{-}1)
$$

The constant polynomial form augmented matrix of these equations is

$$
\begin{bmatrix} 1 & 2 & 0 & 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 4 & 4 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 & -1 & -2 \end{bmatrix} \quad (5.6\text{-}2)
$$

The matrix is reduced by the technique of square elimination from the right. Subtracting the third row from the second row and shifting the resulting second row two places right yields

$$
\begin{bmatrix} 1 & 2 & 0 & 1 & 3 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 & -2 \end{bmatrix} \quad (5.6\text{-}3)
$$

Next subtract the second row from the first, and shift the resulting first row right by two places, yielding

$$
\begin{bmatrix} 0 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 & -2 \end{bmatrix} \quad (5.6\text{-}4)
$$

Add the third row to the first, and shift the resulting first row right by two places. Also add the second row to the third, and shift the resulting third row right by one place. These operations yield

$$
\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 \end{bmatrix} \quad (5.6\text{-}5)
$$

Finally subtracting any row from the other two yields

$$
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 \end{bmatrix} \quad (5.6\text{-}6)
$$

The reduction is now complete, since the rank of (5.6-6) is one. The degree of (5.6-6) is two, indicating that two common roots exist, namely $x=-1$ and $x=-2$, which are the solutions of

$$
x^2 + 3x + 2 = 0 \quad (5.6\text{-}7)
$$

It also follows that $x^2 + 3x + 2$ is the highest common divisor of the polynomials f_1, f_2 and f_3.

The example clearly illustrates the superiority of polynomial form matrix techniques in comparison with the usual classical methods. Sylvester's determinant in this example would be 13×13 and would only determine whether common solutions exist; it would not yield the common roots. The common divisor method, i.e., successive division of one polynomial by another leading to the highest common divisor, is very tedious, especially for more than two polynomials.

5.7. Polynomial matrices of one variable

The polynomial matrix reduction techniques of square and cross elimination discussed in Sections 5.3 and 5.4, are very effective for solving sets of equations containing two variables which can be written in polynomial form. Sections 5.7.1, 5.7.2 and 5.7.3 treat the usual case of polynomial form matrices of one variable which contain two rows. In Sections 5.7.1 and 5.7.2 special formulas are developed for quadratic and cubic polynomial form matrices. Section 5.7.3 discusses the properties of higher order polynomial form matrices. Finally, Section 5.7.4 deals with the problem of polynomial form matrices of one variable which contain more than two rows.

It should be noted that it is of course not possible to reduce all polynomial form augmented matrices. Those matrices which are not reducible have an unconditional rank of one, which means that the matrix is equivalent to a single equation. Hence further reduction of degree or rank is neither possible nor desirable.

In particular, complete reducibility is defined as follows.

Definition

A polynomial form augmented matrix A^m is said to be completely reducible if its degree m can be reduced by using identical row and row shifting operations until there results a matrix $A1'$ which contains only one row of the degree one and all other rows of degree zero.

If a matrix is completely reducible, this implies that non-singular transformation matrices must exist for carrying out the reduction.

Two necessary definitions are as follows.

Definition

Let A^m be a polynomial form augmented matrix of degree m with respect to a variable x_k. Let A^r of degree r $(0 < r < m)$ be any polynomial matrix obtained from A^m by row and row shifting operations. The equation corresponding to any row of A^r is said to be a resultant of A^m. The highest power of the variable x_k appearing in that resultant is called the degree of the resultant.

Definition

If any variable or variables take on the same value or set of values in k solution sets, then it is said that a k-fold repeating solution set exists with respect to those variables. Occasionally k may become infinitely many, and it is said that an infinite-fold repeating solution set exists.

5.7.1. Quadratic reduction

A special formula is readily developed for the solution of a second degree polynomial form matrix of two rows. The following derivation is based on the assumption that the matrix is completely reducible. Let

$$\left.\begin{array}{l} g_{11}(y)x^2 + g_{12}(y)x + g_{13}(y) = 0 \\ g_{21}(y)x^2 + g_{22}(y)x + g_{23}(y) = 0 \end{array}\right] \qquad (5.7.1\text{-}1)$$

be two equations in two variables. The corresponding polynomial form augmented matrix A is

$$A = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{bmatrix} \qquad (5.7.1\text{-}2)$$

The technique of square elimination from the left may be used to solve this matrix in three operations. First, $P_1 A \sim A'$ or

$$\begin{bmatrix} 1 & 0 \\ -g_{21} & g_{11} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{bmatrix}$$

$$\sim \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ 0 & g_{22}^{[1]} & g_{23}^{[1]} \end{bmatrix} \qquad (5.7.1\text{-}3)$$

where

$$g_{22}^{[1]} = g_{11}g_{22} - g_{12}g_{21}$$

$$g_{23}^{[1]} = g_{11}g_{23} - g_{13}g_{21}$$

Second, $(s2r1 \;\; P_2 \;\; s2l1)\, A' \sim A^{[1]}$, or

$$s2r1 \begin{bmatrix} -g_{22}^{[1]} & g_{11} \\ 0 & 1 \end{bmatrix} s2l1 \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ 0 & g_{22}^{[1]} & g_{23}^{[1]} \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & g_{12}^{[1]} & g_{13}^{[1]} \\ 0 & g_{22}^{[1]} & g_{23}^{[1]} \end{bmatrix} \qquad (5.7.1\text{-}4)$$

where

$$g_{12}^{[1]} = g_{11}g_{23}^{[1]} - g_{12}g_{22}^{[1]}$$

$$g_{13}^{[1]} = -g_{13}g_{22}^{[1]}$$

Third, $P_3 A^{[1]} \sim A^{[1]\prime}$, or

$$\begin{bmatrix} g_{22}^{[1]} & -g_{12}^{[1]} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & g_{12}^{[1]} & g_{13}^{[1]} \\ 0 & g_{22}^{[1]} & g_{23}^{[1]} \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & g_{13}^{[2]} \\ 0 & g_{22}^{[1]} & g_{23}^{[1]} \end{bmatrix} \qquad (5.7.1\text{-}5)$$

where

$$g_{13}^{[2]} = g_{13}^{[1]} g_{22}^{[1]} - g_{12}^{[1]} g_{23}^{[1]}$$

Combining these three operations yields the following statement of equivalence

$$A \underset{[|P_1| \;|P_2| \;|P_3| = 0]}{\underset{\sim}{\subset}} P_3 s2r1 \; P_2 \; s2l1 \; P_1 A \qquad (5.7.1\text{-}6)$$

i.e.,

$$[g_{11}(g_{11}g_{22} - g_{12}g_{21}) = 0]$$

All of the solution sets of the original problem (5.7.1-1) must satisfy the resultants of degree zero and one obtained in (5.7.1-5), i.e.,

$$g_{13}^{[2]} = g_{11} \begin{bmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{bmatrix}^2 - g_{11} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{bmatrix} = 0 \qquad (5.7.1.\text{-}7)$$

$$g_{22}^{[1]}x + g_{23}^{[1]} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} x + \begin{bmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{bmatrix} = 0 \qquad (5.7.1\text{-}8)$$

Any additional solution sets, if present, must satisfy

$$g_{11} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = 0 \qquad (5.7.1\text{-}9)$$

Since any value of y which satisfies $g_{11} = 0$ is a solution of (5.7.1-7) and (5.7.1-9), it is not known whether such values are solution sets of the original problem or additional solution sets which were introduced by square elimination.

Further information about the existence of additional roots is obtained by using cross elimination. Two operations are required.

First, $s2r1 \;\; P_1 \;\; A \sim A^{[1]}$, or

$$s2r1 \begin{bmatrix} -g_{21} & g_{11} \\ -g_{23} & g_{13} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{bmatrix} \sim \begin{bmatrix} 0 & g_{12}^{[1]} & g_{13}^{[1]} \\ 0 & g_{22}^{[1]} & g_{23}^{[1]} \end{bmatrix} \qquad (5.7.1\text{-}10)$$

where

$$g_{12}^{[1]} = g_{11}g_{22} - g_{12}g_{21}$$

$$g_{13}^{[1]} = g_{11}g_{23} - g_{13}g_{21}$$

$$g_{22}^{[1]} = g_{13}g_{21} - g_{11}g_{23}$$

$$g_{23}^{[1]} = g_{13}g_{22} - g_{12}g_{23}$$

Second, $P_2 A^{[1]} \sim A^{[1]'}$

$$\begin{bmatrix} -g_{22}^{[1]} & g_{12}^{[1]} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & g_{12}^{[1]} & g_{13}^{[1]} \\ 0 & g_{22}^{[1]} & g_{23}^{[1]} \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & g_{13}^{[1]} \\ 0 & g_{22}^{[1]} & g_{23}^{[1]} \end{bmatrix} \qquad (5.7.1\text{-}11)$$

where

$$g_{13}^{[1]} = g_{12}^{[1]} g_{23}^{[1]} - g_{13}^{[1]} g_{22}^{[1]}$$

Combining these two operations yields the following statement of equivalence

$$\left.\begin{array}{l} A \underset{[|P_1| \; |\tilde{P}_2| = 0]}{\subset} P_2 \text{ s2r1 } P_1 A \\ \text{i.e., } [(g_{11}g_{23} - g_{13}g_{21}) = 0] \\ \\ A \underset{[x \tilde{=} 0]}{\supset} P_2 \text{ s2r1 } P_1 A \end{array}\right]$$

Thus except for the possible trivial solution $x = 0$, all of the solution sets of the original problem (5.7.1-1) must satisfy the resultants of degree zero and one obtained in (5.7.1-11), i.e.,

$$g_{13}^{[2]} = \begin{bmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{bmatrix}^2 - \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{bmatrix} = 0 \qquad (5.7.1\text{-}12)$$

$$g_{22}^{[1]}x + g_{23}^{[1]} = \begin{bmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{bmatrix} x + \begin{bmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{bmatrix} = 0 \qquad (5.7.1.\text{-}13)$$

Any additional solution sets, if present, must satisfy

$$\begin{bmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{bmatrix} = 0 \qquad (5.7.1\text{-}14)$$

Comparing the results obtained from square elimination and cross elimination indicates that in this case cross elimination is preferred since it does not introduce the additional roots which satisfy $g_{11} = 0$.

The formal solution of (5.7.1-1), except for the possible trivial solution $x = 0$, is given by (5.7.1-12) and (5.7.1-13). Any additional solution sets must satisfy (5.7.1-14). It should be noted that if any solution of (5.7.1-12) also satisfies (5.7.1-14), this implies that the conditional rank of (5.7.1-2) is one or zero, and hence repeating roots may exist. This question will be discussed in general in Section 5.7.3.

Example

The following equations

$$\left.\begin{array}{l} yx^2 + (y^2 + 2y + 2)x + y^3 = 0 \\ x^2 + yx + (y^2 - y - 1) = 0 \end{array}\right]$$

have the associated polynomial form augmented matrix

$$\begin{bmatrix} y & (y^2 + 2y + 2) & +y^3 \\ 1 & y & +(y^2 - y - 1) \end{bmatrix}$$

The resultant of degree zero of this matrix, by equation (5.7.1-12), is

$$\begin{bmatrix} y & y^3 \\ 1 & (y^2-y-1) \end{bmatrix} = \begin{bmatrix} y & (y^2+2y+2) \\ 1 & y \end{bmatrix}\begin{bmatrix} (y^2+2y+2) & y^3 \\ y & (y^2-y-1) \end{bmatrix}$$

which has the numerical roots

$y = 2, \; y = -2/3, \; y = -1, \; y = -1$

From the resultant of degree one, given by equation (5.7.1-13),

$$x = -\frac{\begin{vmatrix} (y^2+2y+2) & y^3 \\ y & (y^2-y-1) \end{vmatrix}}{\begin{vmatrix} y & y^3 \\ 1 & (y^2-y-1) \end{vmatrix}} = \frac{y^3 - y^2 - 4y - 2}{y(y+1)}$$

Substitution of the first and second roots of the resultant of degree zero, namely $y = 2$ and $y = -2/3$, is straightforward and yields the solution sets

$y = 2 \qquad x = -1$

$y = -2/3 \qquad x = 1/3$

since neither of these y values satisfies (5.7.1-14). The remaining y value, namely $y = -1$, does satisfy (5.7.1-14) and therefore may or may not be a solution of the original problem. Any attempt to substitute $y = -1$ into the resultant of degree one is frustrated since both the numerator and denominator are zero. Substituting back into the original augmented matrix reduces the problem to a constant polynomial form matrix of rank 1 and degree 2. By Theorem 5-2, two complex solution sets exist, namely

$y = -1 \qquad x = \frac{1}{2}(1 + i\sqrt{3})$

$y = -1 \qquad x = \frac{1}{2}(1 - i\sqrt{3})$

5.7.2. *Cubic reduction*

Another special formula is readily developed for the solution of cubic polynomial form matrices of two rows. The following derivation is also based on the assumption that the matrix is completely reducible.

Let

$$\left.\begin{array}{l} g_{11}(y)x^3 + g_{12}(y)x^2 + g_{13}(y)x + g_{14}(y) = 0 \\ g_{21}(y)^3 \ + g_{22}(y)x^2 + g_{23}(y)x + g_{24}(y) = 0 \end{array}\right]$$

be two equations in two variables. The corresponding polynomial form augmented matrix is

$$A = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \end{bmatrix}$$

The technique of cross elimination may be used to solve this matrix in three operations.

First, s2r1 $P_1 A \sim A^{[1]}$, or

$$\text{s2r1} \begin{bmatrix} -g_{21} & g_{11} \\ -g_{24} & g_{14} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & g_{12}^{[1]} & g_{13}^{[1]} & g_{14}^{[1]} \\ 0 & g_{22}^{[1]} & g_{23}^{[1]} & g_{24}^{[1]} \end{bmatrix}$$

where

$$g_{12}^{[1]} = g_{11}g_{22} - g_{21}g_{12}$$

$$g_{13}^{[1]} = g_{11}g_{23} - g_{21}g_{13}$$

$$g_{14}^{[1]} = g_{11}g_{24} - g_{21}g_{14}$$

$$g_{22}^{[1]} = g_{14}g_{21} - g_{24}g_{11}$$

$$g_{23}^{[1]} = g_{14}g_{22} - g_{24}g_{12}$$

$$g_{24}^{[1]} = g_{14}g_{23} - g_{24}g_{13}$$

The problem is now reduced to that of quadratic reduction. For the sake of completeness, the reduction is completed.

Second, s2r1 $P_2 A^{[1]} \sim A^{[2]}$

$$\text{s2r1} \begin{bmatrix} -g_{22}^{[1]} & g_{12}^{[1]} \\ -g_{24}^{[1]} & g_{14}^{[1]} \end{bmatrix} \begin{bmatrix} 0 & g_{12}^{[1]} & g_{13}^{[1]} & g_{14}^{[1]} \\ 0 & g_{22}^{[1]} & g_{23}^{[1]} & g_{24}^{[1]} \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & g_{13}^{[2]} & g_{14}^{[2]} \\ 0 & 0 & g_{23}^{[2]} & g_{24}^{[2]} \end{bmatrix}$$

Third, $P_3 A^{[2]} \sim A^{[2]'}$

$$\begin{bmatrix} -g_{23}^{[2]} & g_{13}^{[2]} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & g_{13}^{[2]} & g_{14}^{[2]} \\ 0 & 0 & g_{23}^{[2]} & g_{24}^{[2]} \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & g_{14}^{[3]} \\ 0 & 0 & g_{23}^{[2]} & g_{24}^{[2]} \end{bmatrix}$$

Combining these three operations yields the following statement of equivalence

$$\left.\begin{array}{l} A \underset{[|P_1|\ |P_2|\ |P_3|=0]}{\tilde{\subseteq}} \text{s2r1}\ P_1\ \text{s2r1}\ P_2P_3A \\ \\ A \underset{[x=0]}{\tilde{\supseteq}} \text{s2r1}\ P_1\ \text{s2r1}\ P_2P_3A \end{array}\right]$$

Except $x = 0$, all solution sets of the original problem must satisfy the resultant of degree zero, $g_{14}^{[3]} = 0$, and the resultant of degree one, $g_{23}^{[2]} x + g_{24}^{[2]} = 0$. If the resultant of degree zero is divided by $|P_1|$ and is expressed in terms of elements from the original matrix, there obtains

$$\begin{bmatrix} g_{11} & g_{14} \\ g_{21} & g_{24} \end{bmatrix}^3 + \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} g_{12} & g_{14} \\ g_{22} & g_{24} \end{bmatrix}^2$$

$$+ \begin{bmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{bmatrix}^2 \begin{bmatrix} g_{13} & g_{14} \\ g_{23} & g_{24} \end{bmatrix}$$

$$= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{bmatrix} \begin{bmatrix} g_{13} & g_{14} \\ g_{23} & g_{24} \end{bmatrix}$$

$$+ 2 \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{14} \\ g_{21} & g_{24} \end{bmatrix} \begin{bmatrix} g_{13} & g_{14} \\ g_{23} & g_{24} \end{bmatrix}$$

$$+ \begin{bmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{bmatrix} \begin{bmatrix} g_{11} & g_{14} \\ g_{21} & g_{24} \end{bmatrix} \begin{bmatrix} g_{12} & g_{14} \\ g_{22} & g_{24} \end{bmatrix}$$

The resultant of degree one is clearly

$$\begin{bmatrix} g_{12}^{[1]} & g_{14}^{[1]} \\ g_{22}^{[1]} & g_{24}^{[1]} \end{bmatrix} x + \begin{bmatrix} g_{13}^{[1]} & g_{14}^{[1]} \\ g_{23}^{[1]} & g_{24}^{[1]} \end{bmatrix} = 0$$

Any additional solution sets, if they exist, must satisfy

$|P_2|\,|P_3| = 0$ or $|P_1| = 0$, i.e.,

$$\begin{bmatrix} g_{12}^{[1]} & g_{14}^{[1]} \\ g_{22}^{[1]} & g_{24}^{[1]} \end{bmatrix} = 0 \text{ or } \begin{bmatrix} g_{11} & g_{14} \\ g_{21} & g_{24} \end{bmatrix} = 0.$$

Any missing solution sets, if they exist, must satisfy

$|P_1| = 0$ or $x = 0$

$$\begin{bmatrix} g_{11} & g_{14} \\ g_{21} & g_{24} \end{bmatrix} = 0 \quad \text{or} \quad x = 0.$$

5.7.3. Mth order reduction

Before a general procedure is given for mth order reduction, the term conditionally singular transformation matrix and the term conditionally well reduced matrix are defined.

Definition

If a transformation matrix P contains n variables, say $x_1, x_2, \ldots, x_n$, the transformation matrix P is said to be conditionally singular with respect to a set of values $a_1, a_2, \ldots, a_n$, if and only if $|P(a_1, a_2, \ldots, a_n)| = 0$.

Definition

If a polynomial form augmented matrix A^m contains n variables $x_1, x_2, \ldots, x_n$ and is reduced to a matrix A^r of degree r $(0 \leqslant r \leqslant m)$, A^r is said to be a conditionally well reduced matrix with respect to a set of values $a_1, a_2, \ldots, a_n$, if and only if no transformation matrices which are conditionally singular with respect to the set of values $a_1, a_2, \ldots, a_n$ have been used in the process of reduction.

A general procedure is now given for the solution of mth order polynomial form matrices. Let

$$\left.\begin{aligned} g_{11}(y)x^m + g_{12}(y)x^{m-1} + \ldots + g_{1m}(y) = 0 \\ g_{21}(y)x^m + g_{22}(y)x^{m-1} + \ldots + g_{2m}(y) = 0 \end{aligned}\right] \tag{5.7.3-1}$$

be two equations in two variables. The corresponding polynomial form augmented matrix A^m of degree m is

$$\begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ g_{21} & g_{22} & \cdots & g_{2m} \end{bmatrix} \tag{5.7.3.-2}$$

The procedure for solving this problem is largely based on reduction by means of square and cross elimination. The procedure is described first, and the necessary supporting theory is given in part immediately afterward and in part in Section 5.8.

Procedure

The procedure for solving mth order polynomial form augmented matrices is as follows.

(1) Perform square or cross elimination until either a) the unconditional rank of the matrix becomes one, or b) complete reduction is achieved.

(2) If a) occurs, the original equations have infinitely many j-fold repeating solution sets $(1 \leqslant j \leqslant k)$, where k is the lowest degree of the matrix of rank one, or the original equation may have infinite-fold repeating solution sets. These solutions are obtained by solving the single equation from that matrix; for each value of y $(-\infty < y < +\infty)$ there exist j values of x. If any of the above solution sets cause any of the transformation matrices used in the reduction to become conditionally singular, there may of course be additional solution sets. The manner of dealing with these possible additional solution sets is explained in the latter half of the next step.

(3) If b) occurs, the original equations may have one or more j-fold $(1 \leqslant j \leqslant m)$ repeating solution sets and/or one or more infinite-fold repeating solution sets. These solution sets are obtained by solving the resultant of degree zero of the fully reduced matrix to obtain a set of y values. The x values corresponding to these y values are obtained by substituting each y value into the transformation matrices used in the reduction. For each value of y which does not make any of the transformation matrices conditionally singular, the corresponding value of x may be obtained from the resultant of degree one of the fully reduced matrix. If one or more transformation matrices are conditionally singular with respect to one of the y values, say γ, select the most reduced matrix A^k which is conditionally well reduced with respect to γ, substitute the value of γ, and determine the resulting conditional rank which may be zero, one, or two.

If the rank is zero, all values of x satisfy the original equations and an infinite-fold repeating solution set exists. If the rank is one, j-fold repeating solution sets $(1 \leqslant j \leqslant k)$ exist, where $k \leqslant m$. The corresponding values of x are then obtained directly from the resultant of the degree k from A^k. If the rank is two, the resulting constant polynomial form matrix is reduced by square and cross elimination to rank one. The corresponding values of x are the solutions of this matrix of rank one, if any solutions exist.

The theory for the case in which complete reduction is achieved (b) is now given. The case in which the unconditional rank of the matrix becomes one (a) is discussed in Section 5.8.

Theorem 5-3

If a completely reducible polynomial form augmented matrix A^m of lowest degree m contains k-fold repeating solution sets with respect to a numerical value of y, say γ, then either $k \leq m$ or $k = \infty$.

Proof

The conditional rank of A^m with respect to γ can only be zero, one, or two. If the rank is zero, any value of x satisfies A^m and an infinite-fold repeating solution set with respect to γ exists. If the rank is one and A^m is of lowest degree k, a k-fold $(1 \leq k \leq m)$ repeating solution set with respect to γ exists. If the rank is two, reduction of the matrix to rank one also reduces the degree to be less than m, hence fewer than m-fold repeating solution sets with respect to γ exist.

Theorem 5-4

If a completely reducible polynomial form augmented matrix A^m of lowest degree m contains $k(k \leq m)$-fold repeating solution sets with respect to a numerical value of y, say γ, and if A^k is conditionally well reduced with respect to γ, then

(1) A resultant of degree zero obtained by reducing A^m with either square or cross elimination must possess the root γ in at least multiplicity k.

(2) The k values of x corresponding to γ are the roots of any resultant of degree k obtained from the matrix A^k of lowest degree k.

Proof

The second part of this theorem is proved first. The conditional rank of A^k with respect to γ is either zero, one or two. Rank zero indicates that A^k has an infinite-fold repeating solution set with respect to γ. Since A^m is conditionally equivalent to A^k for $y = \gamma$, A^m must also have an infinite-fold repeating solution set with respect to γ. But this contradicts the original premise that A^m has k-fold repeating solution sets. Hence zero rank is impossible. Rank two indicates that A^k has fewer than k-fold repeating solution sets with respect to γ, since by Theorem 5-1 some reduction of degree is always possible before rank one is obtained. Since A^m is conditionally equivalent to A^k for $y = \gamma$, A^m must also have fewer than k-fold repeating solution sets with respect to γ. This also contradicts the original premise that A^m has k-fold repeating solution sets. Hence a rank of two is impossible. Therefore the conditional rank of A^k with respect to γ must be one.
Conversely, if the conditional rank of A^k with respect to γ is one, the k-fold repeating solution set of A^k must also be a k-fold repeating solution set of A^m, since the two matrices are conditionally equivalent.
The first part of the theorem is proved next. Let A^k be the reduced matrix of degree k obtained from A^m by square or cross elimination

$$A^k = \begin{bmatrix} g_{11}^{[k]} & g_{12}^{[k]} & \cdots & g_{1\,k+1}^{[k]} \\ g_{21}^{[k]} & g_{22}^{[k]} & \cdots & g_{2\,k+1}^{[k]} \end{bmatrix}$$

A^k can be reduced to A^{k-1} by premultiplying by the transformation matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}^{-1} \begin{bmatrix} -g_{21}^{[k]} & g_{11}^{[k]} \\ -g_{2\,k+1}^{[k]} & g_{1\,k+1}^{[k]} \end{bmatrix}$$

which perform cross elimination. The proof could also be made performing square elimination. The result of the premultiplication is

$$A^k \underset{\sim}{\subset} \left[\begin{vmatrix} -g_{21}^{[k]} & g_{11}^{[k]} \\ -g_{2\,k+1}^{[k]} & g_{1\,k+1}^{[k]} \end{vmatrix} = 0 \right] A^{k-1} = \begin{bmatrix} g_{11}^{[k-1]} & g_{12}^{[k-1]} & \cdots & g_{1\,k}^{[k-1]} \\ g_{21}^{[k-1]} & g_{22}^{[k-1]} & \cdots & g_{2\,k}^{[k-1]} \end{bmatrix}$$

$$A^k \underset{[x=0]}{\underset{\sim}{\supset}} A^{k-1}$$

where

$$\begin{aligned}
g_{11}^{[k-1]} &= g_{11}^{[k]} g_{22}^{[k]} - g_{21}^{[k]} g_{12}^{[k]} \\
g_{12}^{[k-1]} &= g_{11}^{[k]} g_{23}^{[k]} - g_{21}^{[k]} g_{13}^{[k]} \\
&\vdots \\
g_{1\,k}^{[k-1]} &= g_{11}^{[k]} g_{2\,k+1}^{[k]} - g_{21}^{[k]} g_{1\,k+1}^{[k]} \\
g_{21}^{[k-1]} &= g_{2\,k+1}^{[k]} g_{11}^{[k]} - g_{1\,k+1}^{[k]} g_{21}^{[k]} \\
g_{22}^{[k-1]} &= g_{2\,k+1}^{[k]} g_{12}^{[k]} - g_{1\,k+1}^{[k]} g_{22}^{[k]} \\
&\vdots \\
g_{2\,k}^{[k-1]} &= g_{2\,k+1}^{[k]} g_{1\,k}^{[k]} - g_{1\,k+1}^{[k]} g_{2\,k}^{[k]}
\end{aligned}$$

Since A^m has a k-fold repeating solution set with respect to γ, the conditional rank of A^k must be one, i.e., either the elements of one row vector of A^k are conditionally proportional to the corresponding elements of the other row vector, or all of the elements of one row vector are conditionally zero. In either case all of the elements of A^{k-1} are therefore conditionally zero with respect to γ, and it is certain that each element of A^{k-1} contains the factor $(y - \gamma)$ to at least the first power. Continuing this same line of reasoning, each element of A^{k-2} contains the factor $(y - \gamma)$ to at least the second power. Finally, each element of A^0 contains the factor $(y - \gamma)$ to at least the kth power.

Corollary

If a completely reducible polynomial form augmented matrix A^m of lowest degree m contains an infinite-fold repeating solution set with respect to a numerical value of y, say γ, then

1) The conditional rank of A^m with respect to γ is zero
2) Any resultant of degree zero obtained by either cross elimination or square elimination must possess the root γ in a multiplicity greater than m.

The proof of this corollary is similar to the proof of Theorem 5-4.

Corollary

If A^m is completely reduced by cross elimination or square elimination, and the resultant of degree zero of the completely reduced matrix contains a root γ of multiplicity r, then

1) r-k of these roots are additional roots if there exists a conditionally well reduced matrix with respect to γ, A^k, of degree k.
2) All r of these roots are additional roots if the most reduced matrix which is conditionally well reduced with respect to γ, is or can be conditionally reduced to an augmented matrix whose conditional rank is not equal to the conditional rank of its coefficient matrix.

Proof

1) If there exists a conditionally well reduced matrix with respect to γ, A^k, whose conditional rank is one, then by Theorem 5.4 the original matrix A^m must have a k-fold repeating solution set with respect to γ, and the resultant of degree zero must contain the root γ in at least multiplicity k, say r. Thus $r \geqslant k$. If $r > k$, then r-k of the γ roots must be additional roots introduced while reducing A^m to $A^{1'}$.
2) Let the most reduced matrix which is conditionally well reduced with respect to γ be denoted by A^k. If after substituting γ into A^k the rank of A^k is not equal to the rank of its coefficient matrix, or if further reduction of A^k produces any augmented matrix $A^{k'}$ whose rank is not equal to the rank of its coefficient matrix, then the two equations corresponding to A^k or $A^{k'}$ are contradictory and have no common solutions. Since $A^m \sim A^k \sim A^{k'}$ for $y = \gamma$, clearly γ is not a solution of A^m.

5.7.4. Reduction with multiple rows

The techniques of cross elimination and square elimination are also useful for solving three or more equations in two unknowns. The polynomial form augmented matrix A^m of degree m formed from these equations and having dimensions $n \times (m + 1)$, where n is the number of equations, may or may not be completely reducible. A discussion of the case where the matrix is not completely reducible is given in Section 5.8. For the case where the polynomial form matrix is completely reducible, n-1 resultants of degree zero and one resultant of degree one will be obtained. The common roots of the n-1 resultants of degree zero provide all of the values of y corresponding to the actual and additional solution sets. Of course, if these n-1 resultants of degree zero have no common roots, the original problem has no solution sets. The n-1 resultants of degree zero form a constant polynomial form matrix, the treatment of which has been described in Section 5.6.

5.8. Polynomial matrices of multiple variables

The problem of solving several equations in several unknowns, where none of the variables appear linearly, may be attacked by repeatedly forming polynomial form augmented matrices and applying cross elimination or square elimination.
In the usual case, all of the matrices are completely reducible. If the first matrix formed contains n rows, then n-1 resultants of degree zero and one resultant of degree one are obtained by reduction of the first matrix. A second polynomial form augmented matrix is formed from the n-1 resultants of degree zero, and elimination is again performed. This process is continued until the last polynomial form augmented matrix, which contains only one variable, is completely reduced. In many problems it is not necessary to perform all of these steps, and short cuts often present themselves, depending upon individual circumstances. If it should occur that any of the matrices encountered during the process of reduction are not reducible, the process automatically terminates and it is neither possible nor necessary to carry out further reduction. In this case, there will be infinitely many multiple solution sets, as expressed by the following theorem.

Theorem 5-5

Let A^m be a polynomial form $n \times (m + 1)$ augmented matrix of degree m containing n-1 variables which is not completely reducible. A^m has among its solution sets infinitely many j-fold repeating solution sets, where $1 \leqslant j \leqslant k$, and k is the lowest degree of the most reduced matrix A^k which can be obtained from A^m. The unconditional rank of A^k is automatically one.

Proof

First, the unconditional rank of A^k must be one, since if it is two, further reduction is possible, which contradicts that A^k is the most reduced matrix which can be obtained from A^m. For any arbitrary numerical set of values of the n-1 variables which appear in A^k, less those sets for which A^k is not conditionally well reduced, $A^m \sim A^k$. Upon substitution of these arbitrary values in A^k, a single polynomial of degree j results, where $0 \leqslant j \leqslant k$. For those arbitrary values for which $1 \leqslant j \leqslant k$, j corresponding values of x are abtained, and hence there are infinitely many j-fold repeating solution sets.

Corollary

Let A^m be a polynomial form $n \times (m + 1)$ augmented matrix of degree m containing n-1 variables. If there exists a set of numerical values for some of these n-1 variables such that after substitution A^m cannot be reduced at all, then A^m has infinite-fold solution sets with respect to the set of numerical values.

Example

The three equations in three variables

$$\left.\begin{aligned} 2y^2z^2 + (y^2+1)(x-y)z + 2(x^2-y) &= 0 \\ (2y-1)z^2 + y^2(x-1)z + (x^2-1) &= 0 \\ yz^2 + (x-y^2)z + (x^2-1) &= 0 \end{aligned}\right]$$

whose polynomial form augmented matrix A is

$$\begin{bmatrix} 2y^2 & (y^2+1)(x-y) & 2(x^2-y) \\ (2y-1) & y^2(x-1) & (x^2-1) \\ y & (x-y^2) & (x^2-1) \end{bmatrix}$$

becomes upon substituting $y = 1$ into A the matrix A'

$$\begin{bmatrix} 2 & 2(x-1) & 2(x^2-1) \\ 1 & (x-1)^{\Delta} & (x^2-1) \\ 1 & (x-1) & (x^2-1) \end{bmatrix}$$

which cannot be further reduced, since A' has an unconditional rank of one. Therefore there exist infinite-fold repeating solution sets with respect to $y = 1$, i.e., $y = 1$ and any values of x, z which satisfy the relation

$$z^2 + (x-1)z + (x^2-1) = 0.$$

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